Non-hermitian models in higher dimensions

Pulak Ranjan Giri*
Theory Division, Saha Institute of Nuclear Physics, 1/AF Bidhannagar, Calcutta 700064, India

(Dated: November 17, 2008)

It is know that PT-symmetric models have real spectra provided the symmetry is not spontaneously broken. Even pseudo-hermitian models have real spectra, which enlarge the the class of non-hermitian models possessing real spectra. We however consider non-hermitian models in higher dimensions which are not necessarily explicit PT-symmetric nor pseudo-hermitian. We show that the models may generate real spectra depending upon the coupling constants of the Hamiltonian. Our models thus further generalize the class of non-hermitian systems, which possess real spectra.

PACS numbers: 03.65.-w

The spectrum of a quantum mechanical system becomes real if the Hamiltonian, H, of the corresponding system is self-adjoint [1]. In terms of mathematics, for a self-adjoint Hamiltonian, the domain for it, $D(H)$, is equal to the domain, $D(H^\dagger)$, of its corresponding adjoint operator, $H^\dagger$. This constraint implies that the Hamiltonian would have to be hermitian, $\int(\psi_1)^\dagger H\psi_2 = \int(H\psi_1)^\dagger\psi_2$, $\forall\psi_1,\psi_2 \in D(H)$. However, in order to have real spectra it is not necessary for the Hamiltonian to be hermitian. For example, it is known that for a non-hermitian system we may have real spectra if the Hamiltonian is PT-symmetric [2] and if the symmetry is not spontaneously broken. Complex eigenvalues may arise if the PT-symmetry is spontaneously broken. Even the the constraint, PT-symmetry, can be relaxed to include larger class of models known as pseudo-hermitian, defined by the relation $H^\dagger = SHS^{-1}$, for an hermitian operator S. For pseudo-hermitian Hamiltonian there exists a corresponding hermitian Hamiltonian which has identical spectrum.

The discussion on non-hermitian quantum mechanics is hugely discussed in one dimension [3, 4, 5, 6, 7, 8, 9, 10, 11, 12]. One can however generalize the non-hermitian models to higher dimensions. For example, the higher dimensional PT-symmetric potentials [13, 14, 15] has been studied, which renders real spectra. In spherical coordinates PT-symmetry is defined by $PTrPT^{-1} = r, PT\theta PT^{-1} = \pi - \theta, PT\phi PT^{-1} = \pi + \phi$, $PTiPT^{-1} = -i$. For a Hamiltonian, H, which is separable in spherical coordinates one can write it as $H = H_r + H_\Omega$, where H_r is the radial part and H_Ω is the remaining angular part of the Hamiltonian. Then the PT-symmetry of the Hamiltonian can be decomposed as $PTHPT^{-1} = PTH_rPT^{-1} + PTH_\Omega PT^{-1}$. Since the radial coordinate r remains invariant under PT-transformation, potential of H_r should be real if it respects the symmetry, $PTH_rPT^{-1} = H_r$. However $PTH_\Omega PT^{-1} = H_\Omega$ implies that the potential $V(\theta,\phi)$ for H_Ω may be complex, but would have to be invariant under PT-transformation, $PTV(\theta,\phi)PT^{-1} = V(\theta,\phi)$.

One can even think about the relaxation of the PT-symmetry to include more potentials in higher dimensions, which may have real spectra. But the question is whether it is necessary for a higher dimensional systems to be at least pseudo-hermitian in order to render real eigenvalues. We in this article focus on this issue by studying some general complex potentials. The answer will not be given but the problem we study will give a first step forward in this direction. One should note that it has been shown in [16] that the Hamiltonian for the Calogero model with complex coupling of the inverse square potential does possess real eigen-value if the coupling is chosen properly. Although the Calogero model is in one-dimension, it can be thought of as a higher dimensional model if the co-ordinates of the N Calogero particles on a line are thought to be co-ordinates of a single particle in higher dimensions. Calogero model with complex coupling is a specific example, which possesses real spectrum. But in a general framework it is also possible to show that a complex potential may possesses real spectra. It can also be recalled that in [17] a scale invariant non-hermitian but PT-symmetric potential $V_{\mathcal{PT}} = \frac{c}{r(r+z)} + \frac{c^*}{r(r-z)}$ was considered, in which an electrically charged particle is moving in the background field of a magnetic monopole. This system can be obtained from the generalized MIC-Kepler model, which is the system of two dyons with the axially symmetric potential $V_{MIC} = \frac{c_1}{r(r+z)} + \frac{c_2}{r(r-z)} - \alpha_s/r + s^2/r^2$, by setting the Coulomb term and the extra inverse square term zero, i.e., $\alpha_s = s^2 = 0$ and and generalizing the two constants c_1 and c_2 to complex conjugate numbers. It was shown that the electrically charged particle possesses bound states.

In this article, we discuss in a generic framework that a model in N-dimensions ($N \geq 2$) interacting with the complex anisotropic inverse square potential

$$V(r,\theta,\phi_i) = \frac{1}{r^2}\left(\alpha + \beta\mathcal{F}(\theta,\phi_i)\right) , \tag{1}$$

where α and β are complex couplings, has real spectrum. The advantage of this potential is that the corresponding Hamiltonian can be separated in radial and angular part. We keep the function $\mathcal{F}(\theta,\phi_i)$ completely general and do not assume any specific form at this stage. Although this

*Electronic address: pulakranjan.giri@saha.ac.in

potential is general, it is not completely arbitrary. Due to the form of the potential it makes the Hamiltonian scale invariant. Note however that our potential (1) is complex and may not possess PT-symmetry in general. Then the one particle Hamiltonian reads

$$H = -\frac{\hbar^2}{2\mu}\nabla + \frac{1}{r^2}\left(\alpha + \beta\mathcal{F}(\theta,\phi_i)\right) . \tag{2}$$

Here μ is the reduced mass of the particle. Note that although (2) is written in a general form, it can be related to the electron polar molecule system with the dipolar molecules are taken to be point like. For $\alpha = 0$, $\beta = eD$(real) and $\mathcal{F}(\theta,\phi_i) = \cos\theta$, (2) reduces to the Hamiltonian of electron polar molecule system [18]. In spherical polar coordinates (2) can be separated in radial and angular co-ordinates as

$$H = H_r \oplus r^{-2}H_\Omega . \tag{3}$$

Due to the existence of the complex potential (1), now both the Hamiltonians

$$H_r = -\frac{h^2}{2\mu}\frac{d^2}{dr^2} - \frac{h^2}{2\mu}\frac{N-1}{r}\frac{d}{dr} + \frac{\alpha}{r^2} \tag{4}$$

$$H_\Omega = -\frac{h^2}{2\mu}\nabla_\Omega + \beta\mathcal{F}(\theta,\phi_i) \tag{5}$$

are non-hermitian and possess no PT-symmetry. Here the Hilbert space in N-dimensions, $L^2(\mathbb{R}^N, r^{N-1}drd\Omega)$, has been separated in two orthogonal spaces as $L^2(\mathbb{R}^+, r^{N-1}dr) \otimes L^2(S^{N-1}, d\Omega)$. The eigenvalue equation $H\Psi = E\Psi$ now can be written as two eigenvalue equations in mutually orthogonal Hilbert spaces

$$H_{r,\eta}R_{E,\eta}(r) = ER_{E,\eta}(r), \tag{6}$$

$$H_\Omega Y_\eta(\theta,\phi_i) = \eta Y_\eta(\theta,\phi_i) , \tag{7}$$

where

$$H_{r,\eta} = -\frac{h^2}{2\mu}\frac{d^2}{dr^2} - \frac{h^2}{2\mu}\frac{N-1}{r}\frac{d}{dr} + \frac{(\alpha+\eta)}{r^2} . \tag{8}$$

Since the eigenvalue equation for H_Ω is non-hermitian and does not possess any constraint like PT-symmetry or pseudo-hermiticity, its eigenvalue η will be complex in general and depends on the complex coupling constant β. Suppose $\eta = \eta_R + i\eta_I$ and $\alpha = \alpha_R + i\alpha_I$. We now choose a specific form of the coupling constant α such that $\alpha_I = -\eta_I$. Then the two complex terms of the inverse square potential cancels each other and only the real terms survive, $(\eta_R + \alpha_R)r^{-2}$. The effective radial Hamiltonian

$$H_{r,\eta_R} = -\frac{h^2}{2\mu}\frac{d^2}{dr^2} - \frac{h^2}{2\mu}\frac{N-1}{r}\frac{d}{dr} + \frac{(\alpha_R+\eta_R)}{r^2} , \tag{9}$$

is now hermitian and does possess real eigenvalue but whether it has bound states or not depends on the ranges of the coupling constant $\alpha_R + \eta_R$. The solution is however well known [19]. We give a brief discussion here for completeness. One can now use the transformation $R_{E,\eta}(r) = r^{-(N-1)/2}\chi_{E,\eta}(r)$ on the Schrödinger eigenvalue equation $H_{r,\eta_R}R_{E,\eta}(r) = ER_{E,\eta}(r)$. The Hamiltonian of the transformed eigenvalue equation $\mathcal{H}_{r,\eta_R}\chi_{E,\eta}(r) = E\chi_{E,\eta}(r)$ has the very familiar form $\mathcal{H}_{r,\eta_R} = -\partial_r^2 + \mathrm{g}/r^2$, with $\mathrm{g} = l(l+N-2) + \alpha_R + \eta_R + (N-1)(N-3)/4$. It can be shown that $\mathcal{H}_{r,\eta_R}$ have only one bound state for $-1/4 \le \mathrm{g} < 3/4$. The bound state solutions can be found from the self-adjoint domain

$$\mathcal{D}^\Sigma(L^{-2}) = \{\mathcal{D} + \psi(r, L^{-2}, \Sigma) | \psi(r, L^{-2}, \Sigma) \in \mathcal{D}\} , \tag{10}$$

where $\mathcal{D} = \{\psi(r) \in \mathcal{L}^2(r^{N-1}dr), \psi(0) = \psi'(0) = 0\}$ and $\mathcal{D}$ is the domain of the adjoint Hamiltonian $\mathcal{H}_{r,\eta_R}{}^*$. The length scale L in the domain $\mathcal{D}^\Sigma(L^{-2})$ comes from the self-adjoint extension mechanism. The energy eigenvalue is

$$\begin{aligned} E &= -L^{-2}\mathcal{F}_1(\Sigma), \quad g \neq -1/4 , \\ &= -L^{-2}\mathcal{F}_2(\Sigma), \quad g = -1/4 . \end{aligned} \tag{11}$$

The exact form of the functions $\mathcal{F}_{1,2}(\Sigma)$ can be found from the matching condition of the eigen-function with the domain $\mathcal{D}^\Sigma(L^{-2})$. The bound state eigenfunction is of the form [19]

$$\begin{aligned} R_{E,\eta}(r) &\equiv K_{\sqrt{g+1/4}}(\sqrt{E}r), \quad g \neq -1/4 , \\ &\equiv K_0(\sqrt{E}r), \quad g = -1/4 , \end{aligned} \tag{12}$$

where Ks are modified Bessel function. For more attractive coupling constant $g < -1/4$, the system does possess infinitely many bound states and goes up to negative infinity. For more positive coupling constant $g \ge 3/4$ the system is essentially self-adjoint and does not possess bound state. We now move to some specific examples to justify the above discussion.

2-dimensional model: We consider a 2-dimensional system with complex non PT-symmetric potential

$$V(r,\phi) = \frac{1}{r^2}\beta\exp(i\phi) . \tag{13}$$

Note that this potential belongs to the class defined in (1) where now $\alpha = 0$ and $\mathcal{F}(\theta,\phi_i) = \exp(i\phi)$ have been chosen. For real β this potential coincides with the potential used in [15]. The angular eigenvalue equation (7) can be solved exactly and it has two independent solutions in terms of modified bessel functions (we set the units $\hbar = 2\mu = 1$ through out)

$$Y_\eta(\phi) \equiv I_{2\sqrt{\eta}}(2\sqrt{\beta}\exp(i\phi/2)) , \tag{14}$$

$$\equiv K_{2\sqrt{\eta}}(2\sqrt{\beta}\exp(i\phi/2)) . \tag{15}$$

To find out the exact solution and the eigenvalue η we use the standard periodicity boundary condition of the wavefunction for the azimuthal variable $Y_\eta(\phi) = Y_\eta(\phi + 2\pi)$, which rejects the second solution K. The solution thus becomes (14) with $\sqrt{\eta}$ being non negative integers. Note that although the potential (13) is non PT-symmetric

and complex it renders real eigenvalues. The radial part of the eigenvalue equation obtained from (6),

$$\left[-\frac{d^2}{dr^2}-\frac{1}{r}\frac{d}{dr}+\frac{\eta}{r^2}\right]R_{E,\eta}(r)=ER_{E,\eta}(r)\,, \qquad (16)$$

is hermitian and it has real eigenvalues once the boundary condition is chosen properly. The solutions can be obtained from (11) and (12). This example is thus a generalization of the class of non-hermitian quantum mechanical systems.

3-dimensional model: For the 3-dimensional case we consider a potential of the form

$$V(r,\theta,\phi)=\frac{1}{r^2}\left(\alpha+\beta\cot^2\theta+\gamma\exp(i\phi)\right)\,. \qquad (17)$$

Note that this type of potential with $\gamma=0$ can be found in [20], but the difference is we have made all the coupling constants complex. The eigenvalue problem for the angular variable is now

$$\left[-\frac{1}{\sin\theta}\left(\sin\theta\frac{d}{d\theta}\right)+\frac{\beta\cos^2\theta+m^2}{\sin^2\theta}\right]Y_\eta=\eta Y_\eta\,. \quad (18)$$

This equation is exactly solvable. For real β and $\gamma=0$, it has been solved in [20] with eigenvalue η being real. For complex coupling constants the solutions can be written as

$$Y_\eta(\theta,\phi)\equiv P_\mu^\nu(\cos\theta)I_{2m}(2\sqrt{\gamma}\exp(i\phi/2))\,, \qquad (19)$$

where $\nu=\sqrt{\beta+m^2}$ and $\mu(\mu+1)=\eta+\beta$, but the eigenvalue η is now complex in general. P_μ^ν is the legerdre function and I_{2m} is the modified bessel function. This time the radial Hamiltonian is non-hermitian due to the complex potential $(\alpha+\eta)r^{-2}$. But as mentioned before we can choose the complex part of the coupling α such that $\alpha_I=-\eta_I$. This will make the potential $(\alpha+\eta)r^{-2}=(\alpha_R+\eta_R)r^{-2}$ real. The effective radial eigenvalue equation,

$$\left[-\frac{d^2}{dr^2}-\frac{2}{r}\frac{d}{dr}+\frac{\alpha_R+\eta_R}{r^2}\right]R_{E,\eta}(r)=ER_{E,\eta}(r)\,, \ (20)$$

thus becomes hermitian and its solution can be obtained from (11) and (12). Note that it is also an example of non-hermitian model which generalizes the class of non-hermitian quantum mechanical systems admitting real eigenvalues.

In conclusion, we discussed non-hermitian models in higher diemsnions. The problems discussed are not necessarily PT-symmetric nor explicitly pseudo-hermitian. Exploiting the complex eigen-values of the angular Hamiltonian we managed to cancel the complex potential of the radial Hamiltonian, which makes it hermitian. Although in order to keep the eigenvalue real a Hamiltonian must be pseudo-hermitian at least, in our discussion the higher dimensional Hamiltonians are not explicitly so, still giving real spectra. More efforts are needed to answer the question, raised above, that whether a system should have to be at least pseudo-hermitian in order to get the spectra real. However it may be possible that for a Hamiltonian H, which is non PT-symmetric and non pseudo-hermitian and thus having complex eigen-value E_R+iE_I, one can construct a Hamiltonian $H-iE_I$, which will become PT-symmetric or pseudo-hermitian and therefore will generate real spectra.

[1] M. Reed and B. Simon, *Fourier Analysis, Self-Adjointness* (New York :Academic, 1975).
[2] C. M. Bender and S. Boettcher, Phys. Rev. Lett. **80**, 5243 (1998).
[3] C. M. Bender, S. Boettcher, and P. N. Meisinger, J. Math. Phys. **40**, 2201 (1999).
[4] C. M. Bender, D. C. Brody, and H. F. Jones, Phys. Rev. Lett. **89**, 270401 (2002).
[5] C. M. Bender, D. C. Brody, and H. F. Jones, Am. J. Phys. **71**, 1095 (2003).
[6] A. Sinha, P. Roy, Czech. J. Phys. **54**, 129 (2004).
[7] P. Dorey, C. Dunning and R. Tateo, J. Phys. **A34**, L391 (2001).
[8] P. Dorey, C. Dunning and R. Tateo, J. Phys. **A34**, 5679 (2001).
[9] C. M. Bender, D. C. Brody, L. P. Hughston and B. K. Meister, arXiv:0704.2959.
[10] A. Mostafazadeh, J. Math. Phys. **44**, 974 (2003).
[11] A. Mostafazadeh, J. Math. Phys. **45**, 932 (2004).
[12] A. Mostafazadeh, J.Phys. **A36**, 7081 (2003).
[13] G. Levai, J. Phys. **A40**, F273 (2007).
[14] A. Nanayakkara, Phy. Lett. **A304**, 67 (2002).
[15] S. H. Mazharimousavi, J. Phys. **A41**, 244016 (2008).
[16] P. K. Ghosh and K. S. Gupta, Phys. Lett. **A323**, 29 (2004).
[17] P. R. Giri, arXiv:0805.2827v1 [hep-th].
[18] P. R. Giri, K. S. Gupta, S. Meljanac, and A. Samsarov, Phys. Lett. **A372**, 2967 (2008).
[19] P. R. Giri, arXiv:0708.0707v1 [hep-th].
[20] S. H. Dong, G. H. Sun, and M. Lozada-Cassou, Phys. Lett. **A340**, 94 (2005).